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IKEBUKURO TOKYO
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On the Mordell-Weil Lattice of the Elliptic Curve $y^2 = x^3 + t^m + 1$. II

by

Hisashi USUI

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1. Introduction

In [U], we consider the Mordell-Weil lattice of the elliptic curve

$$E^{(m)} : y^2 = x^3 + t^m + 1$$

defined over $K = k(t)$, where k is an algebraically closed field of characteristic 0. We denote by L_m the Mordell-Weil lattice $E^{(m)}(K)$. The main result of [U] is that all L_m are described by L_9, L_{12}, L_{18}, L_{24}, L_{30}, L_{60} and well known root lattices. In this paper, we study the lattices L_9, L_{12}, L_{18}, L_{24}, L_{30} and L_{60}. We use

$$y^2 = x^3 + t^m - 1$$

in place of

$$y^2 = x^3 + t^m + 1\,.$$

Since k is an algebraically closed field, the lattice structures remain unchanged. So we use the same notation $E^{(m)}$ and L_m. The Kodaira-Néron model $f : S_m \to \mathbb{P}^1$ has a singular fibre of type II at $t = 1$. So we can consider the specialization homomorphism at $t = 1$

$$\mathrm{sp}_1 : E^{(m)}(K) \to f^{-1}(1)^\sharp \cong \boldsymbol{G}_a\,,$$

which maps each K-rational point P of $E^{(m)}$ to the unique intersection point of the section (P) and the fibre $f^{-1}(1)$. In the above, $f^{-1}(1)^\sharp$ is the smooth part of $f^{-1}(1)$, which has a natural structure of algebraic group over k ([S4]).

Let $d = 6m/(m, 6)$. If $p \equiv 1 \bmod d$, then the rank of $E^{(m)}(\overline{\mathbb{F}_p}(t))$ is equal to the rank of $E^{(m)}(k(t))$ ([S1]). According to Shioda, it is known that in this case $E^{(m)}(\overline{\mathbb{F}_p}(t))$ and $E^{(m)}(k(t))$ are isomorphic as lattices. (Later we refer to this fact as $(*)$). We denote the lattice $E^{(m)}(\overline{\mathbb{F}_p}(t))$ by $L_m^{(p)}$.

On the other hand, if $E^{(m)}(\mathbb{F}_p(t))$ is a sublattice of finite index in $E^{(m)}(\overline{\mathbb{F}_p}(t))$, then $E^{(m)}(\mathbb{F}_p(t)) = E^{(m)}(\overline{\mathbb{F}_p}(t))$ (Lemma 2.1). We call a prime number p such that $E^{(m)}(\mathbb{F}_p(t)) = E^{(m)}(\overline{\mathbb{F}_p}(t))$, a *splitting prime number* of $E^{(m)}$. To find a splitting prime number of $E^{(m)}$, we consider a sublattice of finite index in L_m.

Let C_n be the elliptic curve defined by

$$C_n : y^2 = x^3 + t^{n+1} - t\,.$$

Then

$$M = \{(-t^4 x(1/t^3), \sqrt{-1}t^6 y(1/t^3)) \mid (x(t), y(t)) \in C_3(k(t))\}$$

and

$$M_n = \{(x(t^6)/t^2, y(t^6)/t^3) \mid (x(t), y(t)) \in C_n(k(t))\}$$

are sublattices of L_9 and L_{6n} respectively. We give sublattices of finite index as in the following table:

lattice	sublattice	rank
L_9	$L_3(3) + M$	10
L_{12}	$L_4(3) + L_6(2) + M_2$	16
L_{18}	$L_6(3) + L_9(2) + M_3$	20
L_{24}	$L_{12}(2) + M_4$	24
L_{30}	$L_6(5) + L_5(6) + M_5$	24
L_{60}	$L_{12}(5) + L_{30}(2) + M_{10}$	48

The lattice L_{12} has been studied by Shioda [S8] as an example of Mordell-Weil lattices of certain K3 surfaces introduced by Kuwata [K]. See also [CMT].

Let E be one of the elliptic curves

$$E^{(3)},\ C_2\ (D_4 \text{ type}),\quad E^{(4)},\ C_3\ (E_6 \text{ type}),\quad E^{(5)},\ E^{(6)},\ C_4\ (E_8 \text{ type}).$$

Since the associated elliptic surface of E is a rational surface, we have $E(k(t)) \cong E(\overline{\mathbb{F}_p}(t))$ ([O-S]). For E_6 type and E_8 type, Shioda gave the universal polynomial of E explicitly in [S6] and [S7]. The universal polynomial of E is defined by

$$\Phi_E(X) = \prod_{P \in I}(X - \mathrm{sp}(P))\,.$$

Here I is the set of minimal sections and sp is the specialization homomorphism at $t = 1$ for $E = E^{(6)}$ and $t = \infty$ for the other cases ([S4], [S5], [S6]). We call the equation

$$\Phi_E(u) = 0$$

the *fundamental algebraic equation* of E. Shioda showed the following Theorem:

THEOREM 1 ([S4, Theorem (E_8), Theorem (E_6) and Remark in §4])

(i) *Let $E = E^{(5)}, E^{(6)}$ or C_4 and let p be a prime number different from* $\{2, \ldots, 19, 41, 61, 199\}$. *$E(\overline{\mathbb{F}_p}(t))$ has generators of the form $P_i = (g_i t^2 + a_i t + b_i, h_i t^3 + c_i t^2 + d_i t + e_i)$ $(i = 1, \ldots, 8)$, which are minimal sections. If we let $u_i = \mathrm{sp}(P_i)$, then $g_i, a_i, b_i, h_i, c_i, d_i, e_i \in \mathbb{F}_p[u_1, \ldots, u_8][u_i^{-1}]$.*

(ii) *Let $E = E^{(4)}$ or C_3 and let p be a prime number different from* $\{2, \ldots, 7\}$. *$E(\overline{\mathbb{F}_p}(t))$ has generators of the form $P_i = (a_i t + b_i, t^2 + d_i t + e_i)$ $(i = 1, \ldots, 6)$, which are minimal sections. If we let $u_i = \mathrm{sp}(P_i)$, then $a_i, b_i, d_i, e_i \in \mathbb{F}_p[u_1, \ldots, u_6]$.*

So if all the roots of the fundamental algebraic equation of E belong to $\mathbb{F}_p$ and if p is different from a few primes as in this theorem, then we have $E(\mathbb{F}_p(t)) = E(\overline{\mathbb{F}_p}(t))$.

For D_4 type, $E^{(3)}$ and C_2 are slightly different from the elliptic curves considered in [S4]. But these are essentially the same. So we study following [S4] and have necessary results.

Then we can find a splitting prime number of $E^{(m)}$ except for the case $m = 60$. We show the outline for the case $m = 9$, for example. The coefficients of the generators of $L_3(3)$ are the same as that of L_3 and the coefficients of the generators of M are the same as that of C_3 or $\sqrt{-1}$ times of them. So if $E^{(3)}(\mathbb{F}_p(t)) = E^{(3)}(\overline{\mathbb{F}_p}(t))$, $C_3(\mathbb{F}_p(t)) = C_3(\overline{\mathbb{F}_p}(t))$ and if $\sqrt{-1} \in \mathbb{F}_p$, then we have a sublattice of finite index in $E^{(9)}(\overline{\mathbb{F}_p}(t))$. This sublattice is included in $E^{(9)}(\mathbb{F}_p(t))$. By Lemma 2.1, such a p is a splitting prime number of $E^{(9)}$.

For the case $m = 60$, the rank of C_{10} is 16 and the associated elliptic surface is not a rational surface. So we can not find a splitting prime number of $E^{(60)}$ in the same way.

Let p be a splitting prime number of $E^{(m)}$. There are only finitely many possibility of the minimal sections in $E^{(m)}(\mathbb{F}_p(t))$. So we can get all minimal sections. We can also calculate the height pairings and find generators of them. If we can show that the lattice generated by minimal sections is equal to $E^{(m)}(\mathbb{F}_p(t))$, then we get the lattice structure of $L_m^{(p)}$.

In this way, we get the lattice structure of $L_9^{(p)}$ in Section 2 and $L_{12}^{(p)}$ in Section 3. We can also find splitting prime numbers for the cases $m = 18, 24, 30$. But to get the minimal sections for these cases we need a lot of time.

For the general theory of Mordell-Weil lattices, we refer to [S2] and [S5].

We use the same notation as in [U]. We denote by l_m the rank of L_m. We gave an explicit formula for l_m in [U]. We denote by $L(c)$ the lattice whose pairing is c times of the pairing of the lattice L. So we denote the lattice $E^{(ab)}(k(t^b))$ by $L_a(b)$. This is a sublattice of L_{ab}. For sublattices N_1 and N_2 of L, $N_1 + N_2$ is the sublattice of L generated by N_1 and N_2. A_2^*, D_4^*, E_6^* are the dual lattices of the root lattices A_2, D_4, E_6 respectively and E_8 is the unique positive-definite even unimodular lattice of rank 8 (cf. [C-S, Ch. 4]).

We use UBASIC to find minimal sections and calculate height pairings etc. I would like to thank Professor Kida for advising me about UBASIC.

I would like to thank Professor Shioda. He advised me to study the lattice L_{12} and led me to this series of works.

2. The lattice structure of L_9

Let $f : S_9 \to \mathbb{P}^1$ be the Kodaira-Néron model of $E^{(9)}/K$. The numerical invariants of this surface are given in [S3, §3]. It is an elliptic K3 surface and has a reducible fibre of type I_0^* at $t = \infty$:

$$f^{-1}(\infty) = \Theta_0 + \Theta_1 + \Theta_2 + \Theta_3 + 2\Theta_4\,,$$

where Θ_0 meets the zero section. The explicit formula for the height pairing is given in [S2, Theorem 8.6]. Let P and Q be rational points such that the associated sections (P)

and (Q) intersect Θ_i and Θ_j respectively.

$$\langle P, Q\rangle = 2 + (PO) + (QO) - (PQ) - \mathrm{contr}(P, Q)\,. \tag{2.1}$$

$$\langle P, P\rangle = 4 + 2(PO) - \mathrm{contr}(P, P)\,. \tag{2.2}$$

$$\mathrm{contr}(P, Q) = \begin{cases} 1 & \text{if } i = j \neq 0 \\ \dfrac{1}{2} & \text{if } i \neq j \text{ and } ij \neq 0 \\ 0 & \text{if } ij = 0 \end{cases} \tag{2.3}$$

Here, (PO) is the intersection number of the section (P) and the zero section (O) and similarly for (QO), (PQ). The arithmetic genus of S_9 is 2.

LEMMA 2.1. *Let k_0 be a field of arbitrary characteristic and let $\overline{k_0}$ be its algebraic closure. If $E^{(m)}(k_0(t))$ is a sublattice of finite index in $E^{(m)}(\overline{k_0}(t))$, then $E^{(m)}(k_0(t)) = E^{(m)}(\overline{k_0}(t))$.*

Proof. Let $P \in E^{(m)}(\overline{k_0}(t))$ and let $\sigma \in \mathrm{Gal}(\overline{k_0}/k_0)$. There is a natural number n such that $nP \in E^{(m)}(k_0(t))$. For such an n, $(nP)^\sigma = nP$. On the other hand, $(nP)^\sigma = nP^\sigma$. So we have $nP^\sigma = nP$ and $n(P^\sigma - P) = O$. Since $E^{(m)}(\overline{k_0}(t))$ is torsion free ([S3, Proposition 3.7]), we have $P^\sigma - P = O$ and $P^\sigma = P$. This means that $P \in E^{(m)}(k_0(t))$. *q.e.d.*

Let C_3 be the elliptic curve defined by

$$C_3 : y^2 = x^3 + t^4 - t\,.$$

This is the (E_6, q_1)-model in [S7]. In this case $q_1 = -1$. The universal polynomial is given in [S7]. So we have the fundamental algebraic equation:

$$u^{27} + 1344u^{18} - 40704u^9 + 4096 = 0\,. \tag{2.4}$$

If $(x(t), y(t))$ is a $k(t)$-rational point of C_3, then $(-t^4x(1/t^3), \sqrt{-1}t^6y(1/t^3))$ is a $k(t)$-rational point of $E^{(9)}$. So

$$M = \{(-t^4x(1/t^3), \sqrt{-1}t^6y(1/t^3)) \mid (x(t), y(t)) \in C_3(k(t))\}$$

is a sublattice of L_9. We have $M \cong E_6^*(3)$.

LEMMA 2.2. *$L_3(3)$ and M are orthogonal.*

Proof. We define automorphism σ_3 on $E^{(9)}$ by

$$\sigma_3 : t \to \zeta_3 t\,.$$

In the above, ζ_3 is a cubic root of 1. If P is a point of $L_3(3)$ and $Q = (x, y)$ is a point of M,

$$\begin{aligned}\langle P, Q\rangle &= \langle \sigma_3(P), \sigma_3(Q)\rangle = \langle P, \zeta_3 Q\rangle \\ &= \langle \sigma_3^2(P), \sigma_3^2(Q)\rangle = \langle P, \zeta_3^2 Q\rangle\,.\end{aligned}$$

Here $\zeta_3 Q = (\zeta_3 x, y)$ and $\zeta_3^2 Q = (\zeta_3^2 x, y)$. Since $Q + \zeta_3 Q + \zeta_3^2 Q = O$, we have $\langle P, Q\rangle = 0$. This means that $L_3(3)$ and M are orthogonal. *q.e.d.*

LEMMA 2.3. *$L_3(3) + M$ is a sublattice of finite index in L_9.*

Proof. The rank of $L_3(3)$ is $l_3 = 4$ and the rank of M is 6. By Lemma 2.2, the rank of $L_3(3) + M$ is 10. Since the rank of L_9 is $l_9 = 10$, $L_3(3) + M$ is a sublattice of finite index in L_9. *q.e.d.*

Although $E^{(3)}$ is slightly different from the elliptic curve considered in [S4], these are essentially the same. So we study $E^{(3)}$ following [S4]. For details, see [S4].

The associated elliptic surface has a singular fibre of type I_0^* at $t = \infty$. $L_3 = E^{(3)}(k(t)) \cong D_4^*$ ([O-S, Main Theorem No. 9]).

The points of the form:

$$\begin{cases} x = g + a(t-1) \\ y = h + c(t-1) \end{cases} \quad g, a, h, c \in k\,. \tag{2.5}$$

generate L_3. We substitute (2.5) into the defining equation of $E^{(3)}$, and look at the coefficients of $(t-1)^m$ for $m = 0, 1, 2, 3$. Then we have

$$h^2 = g^3 \tag{2.6}$$

$$2hc = 3g^2a + 3 \tag{2.7}$$

$$c^2 = 3ga^2 + 3 \tag{2.8}$$

$$0 = a^3 + 1 \tag{2.9}$$

The points such that $a = -1$ generate a sublattice 1_4 which is isomorphic to the unit matrix of degree four. The index of 1_4 in L_3 is 2. Let $u = \mathrm{sp}_1(P) = g/h$. By (2.6), we have $g = u^{-2}$, $h = u^{-3}$. By (2.7) and (2.8), we have

$$3u^8 - 6u^4 - 4u^2 - 1 = 0\,. \tag{2.10}$$

For roots u of (2.10), letting $g = u^{-2}$, $h = u^{-3}$ and $c = (-3g^2+3)/2h$, we have all points of the form:

$$\begin{cases} x = g - (t-1) \\ y = h + c(t-1) \end{cases}\,. \tag{2.11}$$

The minimal prime number p such that the equations (2.4) and (2.10) can be solved in $\mathbb{F}_p$ is 433. We have the following Proposition:

PROPOSITION 1. $L_9^{(433)} := E^{(9)}(\overline{\mathbb{F}_{433}}(t)) = E^{(9)}(\mathbb{F}_{433}(t))$.

Proof. Let $p = 433$. $\sqrt{-1} = 179$ in $\mathbb{F}_p$. In this case, $d = 6m/(m, 6) = 18$. Since $p \equiv 1 \bmod 18$, by the algorithm of [S1], we have the rank of $L_9^{(p)}$ is equal to $l_9 = 10$.

Since the associated elliptic surfaces of $E^{(3)}$ and C_3 are rational surfaces, we have $L_3^{(p)} \cong L_3$ and $C_3(\overline{\mathbb{F}_p}(t)) \cong C_3(k(t))$. The lattices

$$L_3^{(p)}(3) = \{(x(t^3), y(t^3)) \mid (x(t), y(t)) \in L_3^{(p)}\}$$

and

$$M^{(p)} = \{(-t^4 x(1/t^3), \sqrt{-1}t^6 y(1/t^3)) \mid (x(t), y(t)) \in C_3(\overline{\mathbb{F}_p}(t))\}$$

are sublattices of $L_9^{(p)}$ and they are isomorphic to $L_3(3)$ and M respectively. Then $L_3^{(p)}(3)+M^{(p)}$ is a sublattice of finite index in $L_9^{(p)}$.

All the roots u of (2.10) belong to $\mathbb{F}_p$. Letting $g = u^{-2}$, $h = u^{-3}$ and $c = (-3g^2 + 3)/2h$, we have the points of the form:

$$(g - (t-1), h + c(t-1))\,.$$

They generate the subgroup 1_4 of index 2 in $L_3^{(p)}$. Then the points

$$(g - (t^3 - 1), h + c(t^3 - 1))$$

generate a subgroup of index 2 in $L_3^{(p)}(3)$. These points belong to $E^{(9)}(\mathbb{F}_p(t))$.

All the roots of (2.4) belong to $\mathbb{F}_p$. By Theorem 1, the points of the form:

$$(x_i(t), y_i(t)) = (a_i t + b_i, t^2 + d_i t + e_i) \quad (i = 1, \dots, 6)$$

generate $C_3(\overline{\mathbb{F}_p}(t))$ and $a_i, b_i, d_i, e_i \in \mathbb{F}_p$. Then the points

$$(-t^4(x_i(1/t^3), \sqrt{-1}t^6 y_i(1/t^3)) \quad (i = 1, \dots, 6)$$

generate $M^{(p)}$. These points belong to $E^{(9)}(\mathbb{F}_p(t))$. So we have $M^{(p)} \subset E^{(9)}(\mathbb{F}_p(t))$.

By Lemma 2.1, we have $E^{(9)}(\mathbb{F}_p(t)) = E^{(9)}(\overline{\mathbb{F}_p}(t))$. *q.e.d.*

By (2.2) and (2.3), the minimal number of $\langle P, P\rangle$ is 3 and

$$\langle P, P\rangle = 3 \Leftrightarrow (PO) = 0 \quad \text{and} \quad \mathrm{contr}(P, P) = 1\,.$$

The points $P \in L_9^{(433)}$ such that $\langle P, P\rangle = 3$ are called minimal sections. By the same argument as [S2, Lemma 10.9] and [S4, Lemma 6.1], the minimal sections are of the form:

$$\text{(2.12)} \qquad \begin{cases} x = x_0 + x_1(t-1) + \cdots + x_3(t-1)^3 \\ y = y_0 + y_1(t-1) + \cdots + y_5(t-1)^5 \end{cases} \quad x_0, \dots, x_3, y_0, \dots, y_5 \in \mathbb{F}_{433}\,.$$

We substitute (2.12) into the defining equation of $E^{(9)}$, and look at the coefficients of $(t-1)^m$ for $m = 0, 1, 2, \dots, 12$. Then we have

$$\text{(2.13)} \qquad y_0^2 = x_0^3$$

$$\text{(2.14)} \qquad 2y_0y_1 = 9 + 3x_0^2x_1$$

$$\text{(2.15)} \qquad y_1^2 + 2y_0y_2 = 36 + 3x_0x_1^2 + 3x_0^2x_2$$

$$\text{(2.16)} \qquad 2y_1y_2 + 2y_0y_3 = 84 + x_1^3 + 6x_0x_1x_2 + 3x_0^2x_3$$

$$\text{(2.17)} \qquad y_2^2 + 2y_1y_3 + 2y_0y_4 = 126 + 3x_1^2x_2 + 3x_0x_2^2 + 6x_0x_1x_3$$

$$\text{(2.18)} \qquad 2y_2y_3 + 2y_1y_4 + 2y_0y_5 = 126 + 3x_1x_2^2 + 3x_1^2x_3 + 6x_0x_2x_3$$

$$\text{(2.19)} \qquad y_3^2 + 2y_2y_4 + 2y_1y_5 = 84 + x_2^3 + 6x_1x_2x_3 + 3x_0x_3^2$$

$$\text{(2.20)} \qquad 2y_3y_4 + 2y_2y_5 = 36 + 3x_2^2x_3 + 3x_1x_3^2$$

$$\text{(2.21)} \qquad y_4^2 + 2y_3y_5 = 9 + 3x_2x_3^2$$

$$2y_4y_5 = 1 + x_3^3 \tag{2.22}$$

$$y_5^2 = 0 \tag{2.23}$$

Let $u = \mathrm{sp}_1(P) = x_0/y_0$. By (2.13), we have $x_0 = u^{-2}$ and $y_0 = u^{-3}$. By (2.22) and (2.23), we have $y_5 = 0$ and $x_3^3 + 1 = 0$. Let $x_3 = -1$. By (2.14), (2.15), (2.16) and (2.17), y_1, y_2, y_3 and y_4 are polynomials of u, u^{-1}, x_1 and x_2. We substitute every element of $\mathbb{F}_{433}^{\times}$ into u and every element of $\mathbb{F}_{433}$ into x_1, and check if there is x_2 satisfying the equations (2.18), (2.19), (2.20) and (2.21). Then we can get all minimal sections such that $x_3 = -1$. Multiplying x by 198 or 234, we get all the other minimal sections. The numbers 198 and 234 are the cubic roots of 1 in $\mathbb{F}_{433}$.

In this way, we find 240 minimal sections. Let $L_{9\,\mathrm{min}}^{(433)}$ be the lattice generated by these 240 minimal sections.

We can calculate the height pairings of two minimal sections. Let

$$\begin{aligned} P &= (x(t), y(t)) \\ &= (x_0 + x_1t + x_2t^2 + x_3t^3, y_0 + y_1t + y_2t^2 + y_3t^3 + y_4t^4) \end{aligned}$$

and

$$\begin{aligned} Q &= (z(t), w(t)) \\ &= (z_0 + z_1t + z_2t^2 + z_3t^3, w_0 + w_1t + w_2t^2 + w_3t^3 + w_4t^4) \end{aligned}$$

be two minimal sections. By (2.1), we have

$$\langle P, Q\rangle = 2 - (PQ) - \mathrm{contr}(P, Q)\,.$$

If $x_3 = z_3$, then (P) and (Q) meet the same component of $f^{-1}(\infty)$ and if $x_3 \neq z_3$, then they meet the different components (cf. [S4, Lemma 6.2]). By (2.3), we have

$$\mathrm{contr}(P, Q) = \begin{cases} 1 & \text{if } x_3 = z_3 \\ \dfrac{1}{2} & \text{if } x_3 \neq z_3 \end{cases}\,.$$

The intersection number (PQ) is calculated by

$$\begin{aligned} (PQ) &= \deg(\gcd(x(t) - z(t), y(t) - w(t))) \\ &\quad + \min(3 - \deg(x(t) - z(t)), 4 - \deg(y(t) - w(t)))\,. \end{aligned}$$

Here $\min(a, b)$ is the smaller number of a and b, $\deg(f(t))$ is the degree of polynomial $f(t)$ and $\gcd(f(t), g(t))$ is the greatest common divisor of two polynomials $f(t)$ and $g(t)$. The second term $\min(3 - \deg(x(t) - z(t)), 4 - \deg(y(t) - w(t)))$ is the intersection multiplicity at $t = \infty$.

Using the height pairings, we find generators of $L_{9\,\mathrm{min}}^{(433)}$.

For one minimal section P, there are 6 minimal sections:

$$P\,,\quad \zeta_3 P\,,\quad \zeta_3^2 P\,,\quad -P\,,\quad -\zeta_3 P\,,\quad -\zeta_3^2 P\,.$$

Here $\zeta_3 P$ and $\zeta_3^2 P$ are the same as in the proof of Lemma 2.2. Since $\zeta_3^2 P = -P - \zeta_3 P$, these 6 sections are described by P and $\zeta_3 P$. So we may consider only 80 minimal sections.

We have 4 independent minimal sections in $L_3(3)$. Adding one by one and calculating the determinant of the height pairing matrix, we can find independent 10 sections.

Once we get independent 10 sections, taking the height pairings with them, we can treat all sections as lattice points in $\mathbb{Z}^{10}$. Then, by elimination, we can find generators.

PROPOSITION 2. *$L_{9\min}^{(433)}$ has the following properties:*
There are 240 *minimal sections.*
The following 10 *points generate $L_{9\min}^{(433)}$:*

$$P_1 = (432t^3, 179)$$
$$P_2 = (235t^3, 179)$$
$$P_3 = (432t^3 + 421, 225t^3 + 51)$$
$$P_4 = (235t^3 + 222, 225t^3 + 51)$$
$$P_5 = (432t^3 + 200t^2 + 234t + 421, 274t^4 + 416t^3 + 401t^2 + 257t + 382)$$
$$P_6 = (235t^3 + 197t^2 + t + 222, 274t^4 + 416t^3 + 401t^2 + 257t + 382)$$
$$P_7 = (432t^3 + 108t^2 + 239t + 362, 415t^4 + 50t^3 + 405t^2 + 21t + 354)$$
$$P_8 = (235t^3 + 167t^2 + 125t + 231, 415t^4 + 50t^3 + 405t^2 + 21t + 354)$$
$$P_9 = (432t^3 + 70t^2 + 391t + 273, 155t^4 + 374t^3 + 15t^2 + 119t + 354)$$
$$P_{10} = (235t^3 + 4t^2 + 344t + 362, 155t^4 + 374t^3 + 15t^2 + 119t + 354)$$

The Gram matrix is

$$\begin{pmatrix}
3 & -3/2 & 0 & 3/2 & 0 & 3/2 & -1 & 1/2 & -1 & 1/2 \\
-3/2 & 3 & -3/2 & 0 & -3/2 & 0 & 1/2 & -1 & 1/2 & -1 \\
0 & -3/2 & 3 & -3/2 & 1 & -1/2 & -1 & 1/2 & 0 & 3/2 \\
3/2 & 0 & -3/2 & 3 & -1/2 & 1 & 1/2 & -1 & -3/2 & 0 \\
0 & -3/2 & 1 & -1/2 & 3 & -3/2 & 0 & -1/2 & 1 & 1/2 \\
3/2 & 0 & -1/2 & 1 & -3/2 & 3 & 1/2 & 0 & -3/2 & 1 \\
-1 & 1/2 & -1 & 1/2 & 0 & 1/2 & 3 & -3/2 & 0 & 1/2 \\
1/2 & -1 & 1/2 & -1 & -1/2 & 0 & -3/2 & 3 & -1/2 & 0 \\
-1 & 1/2 & 0 & -3/2 & 1 & -3/2 & 0 & -1/2 & 3 & -3/2 \\
1/2 & -1 & 3/2 & 0 & 1/2 & 1 & 1/2 & 0 & -3/2 & 3
\end{pmatrix}$$

The determinant is $243/4 = 3^5/2^2$.
The center density (*cf.* [C-S]) *is* $3^2\sqrt{3}/2^9 = 0.030446\ldots$.

We have the following theorem.

THEOREM 2. $L_9^{(433)} = L_{9\min}^{(433)}$.

So $L_9^{(433)}$ has the properties given in Proposition 2.

Proof. Since $L_9^{(433)}$ is a half integral lattice of rank 10, the denominator of the determinant of $L_9^{(433)}$ is a divisor of 2^{10}. So if $L_9^{(433)} \neq L_{9\min}^{(433)}$, there is a point $P \in L_9^{(433)}$ such that $P \notin L_{9\min}^{(433)}$ and $2P$ or $3P \in L_{9\min}^{(433)}$.

Let P be a point in $L_9^{(433)}$ such that $P \notin L_{9\min}^{(433)}$ and $3P \in L_{9\min}^{(433)}$. We can assume that

$$3P = a_1 P_1 + \cdots + a_{10} P_{10} \quad (a_i = -1, 0, 1)\,.$$

We can calculate

$$\begin{aligned}\langle P, P_i\rangle &= \frac{1}{3}\langle 3P, P_i\rangle = \frac{1}{3}\langle a_1 P_1 + \cdots + a_{10} P_{10}, P_i\rangle \\ &= \frac{1}{3}(a_1\langle P_1, P_i\rangle + \cdots + a_{10}\langle P_{10}, P_i\rangle)\,.\end{aligned}$$

Since $\langle P, P_i\rangle \in \frac{1}{2}\mathbb{Z}$,

$$2\langle P, P_i\rangle = \frac{1}{3}(a_1 \cdot 2\langle P_1, P_i\rangle + \cdots + a_{10} \cdot 2\langle P_{10}, P_i\rangle) \in \mathbb{Z}\,.$$

Solving the simultaneous equations

$$a_1 \cdot 2\langle P_1, P_i\rangle + \cdots + a_{10} \cdot 2\langle P_{10}, P_i\rangle = 0 \quad (i = 1, 2, \ldots, 10)$$

in $\mathbb{F}_3$, we get the following relations modulo 3:

$$\begin{pmatrix} a_1 \\ a_3 \\ a_5 \\ a_7 \\ a_9 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a_2 \\ a_4 \\ a_6 \\ a_8 \\ a_{10} \end{pmatrix}.$$

We substitute $-1, 0, 1$ into a_2, a_4, a_6, a_8 and a_{10}, and calculate

$$\langle P, P\rangle = \frac{1}{9}\langle 3P, 3P\rangle = \cdots.$$

$\langle P, P\rangle$ must be an integer greater than 3. Only 16 cases in the following table satisfy these conditions. But for each case, Q in the table is a minimal section.

a_2	a_4	a_6	a_8	a_{10}	$\langle P, P\rangle$	Q
−1	−1	−1	1	−1	7	$P - P_5 - P_8$
−1	−1	−1	1	0	4	$P + P_2$
−1	−1	1	1	1	5	$P + P_9$
−1	0	1	1	1	7	$P - P_1 - P_3$
−1	1	1	−1	1	5	$P + P_9$
−1	1	1	1	1	7	$P + P_9$
0	−1	−1	1	−1	7	$P + P_4$
0	0	0	−1	−1	4	$P + P_1$
0	0	0	1	1	4	$P + P_2$
0	1	1	−1	1	7	$P - P_4$
1	−1	−1	−1	−1	7	$P + P_1$
1	−1	−1	1	−1	5	$P + P_4$
1	0	−1	−1	−1	7	$P - P_2 - P_5$
1	1	−1	−1	−1	5	$P + P_1$
1	1	1	−1	0	4	$P + P_3$
1	1	1	−1	1	7	$P - P_2 - P_4$

This contradicts $P \notin L_{9\min}^{(433)}$. So there is no P in $L_9^{(433)}$ such that $P \notin L_{9\min}^{(433)}$ and $3P \in L_{9\min}^{(433)}$.

Next let P be a point in $L_9^{(433)}$ such that $P \notin L_{9\min}^{(433)}$ and $2P \in L_{9\min}^{(433)}$. We can assume that

$$2P = a_1P_1 + \cdots + a_{10}P_{10} \quad (a_i = 0, 1)\,.$$

We can calculate

$$\begin{aligned}\langle P, P_i\rangle &= \frac{1}{2}\langle 2P, P_i\rangle = \frac{1}{2}\langle a_1P_1 + \cdots + a_{10}P_{10}, P_i\rangle \\ &= \frac{1}{2}(a_1\langle P_1, P_i\rangle + \cdots + a_{10}\langle P_{10}, P_i\rangle)\,.\end{aligned}$$

Since $\langle P, P_i\rangle \in \frac{1}{2}\mathbb{Z}$,

$$2\langle P, P_i\rangle = \frac{1}{2}(a_1 \cdot 2\langle P_1, P_i\rangle + \cdots + a_{10} \cdot 2\langle P_{10}, P_i\rangle) \in \mathbb{Z}\,.$$

Solving the simultaneous equations

$$a_1 \cdot 2\langle P_1, P_i\rangle + \cdots + a_{10} \cdot 2\langle P_{10}, P_i\rangle = 0 \quad (i = 1, 2, \ldots, 10)$$

in $\mathbb{F}_2$, we get the following relations modulo 2:

$$\begin{aligned}a_1 &= a_3 + a_5 + a_7 + a_9\,,\\ a_2 &= a_4 + a_6 + a_8 + a_{10}\,.\end{aligned}$$

We substitute 0 or 1 into $a_3, a_4, \ldots, a_{10}$, and calculate

$$\langle P, P\rangle = \frac{1}{4}\langle 2P, 2P\rangle = \cdots .$$

$\langle P, P\rangle$ must be an integer greater than 3. Only 9 cases in the following table satisfy these conditions. But for each case, Q in the table is a minimal section or $\langle Q, Q\rangle = 2$.

a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	$\langle P, P\rangle$	Q	$\langle Q, Q\rangle$
0	1	0	1	1	0	0	0	5	$P - P_1$	3
0	1	1	1	0	1	0	1	6	$P - P_1$	2
1	0	0	1	0	0	0	1	5	$P - P_1$	3
1	0	1	0	1	1	0	1	5	$P + P_2$	2
1	0	1	1	0	0	1	1	5	$P + P_2$	3
1	0	1	1	1	0	0	1	6	$P - P_{10}$	2
1	1	0	1	0	0	0	1	5	$P + P_9$	3
1	1	0	1	0	1	0	1	7	$P - P_1$	3
1	1	1	1	1	1	0	1	7	$P - P_{10}$	3

This contradicts $P \in L_9^{(433)}$ and $P \notin L_{9\min}^{(433)}$. So there is no P in $L_9^{(433)}$ such that $P \notin L_{9\min}^{(433)}$ and $2P \in L_{9\min}^{(433)}$. *q.e.d.*

COROLLARY 1. *If we assume* (∗), *then* L_9 *has the following properties*:
There are 240 *minimal sections of norm* 3.
The Gram matrix is the same as in Proposition 2.
The determinant is 243/4.
The center density is $3^2\sqrt{3}/2^9 = 0.030446\cdots$.

REMARK 1. We can also get all points $P \in E^{(9)}(\mathbb{F}_{433}(t))$ such that $\langle P, P\rangle = 4$. There are 270 such points.

3. The lattice structure of L_{12}

Let $f : S_{12} \to \mathbb{P}^1$ be the Kodaira-Néron model of $E^{(12)}/K$. The numerical invariants of this surface are given in [S3, §3]. It is an elliptic K3 surface and has no reducible fibres. The explicit formula for the height pairing is given in [S2, Theorem 8.6].

$$\langle P, Q\rangle = 2 + (PO) + (QO) - (PQ). \tag{3.1}$$

$$\langle P, P\rangle = 4 + 2(PO). \tag{3.2}$$

Let C_n be the elliptic curve defined by

$$C_n : y^2 = x^3 + t^{n+1} - t.$$

If $(x(t), y(t))$ is a $k(t)$-rational point of C_n, then $(x(t^6)/t^2, y(t^6)/t^3)$ is a $k(t)$-rational point of $E^{(6n)}$. So

$$M_n = \{(x(t^6)/t^2, y(t^6)/t^3) \mid (x(t), y(t)) \in C_n(k(t))\}$$

is a sublattice of L_{6n}.

LEMMA 3.1. *$L_{2n}(3) + L_{3n}(2)$ and M_n are orthogonal.*

Proof. We define automorphisms σ_2 and σ_3 on $E^{(6n)}$ by

$$\sigma_2 : t \to -t$$
$$\sigma_3 : t \to \zeta_3 t\,.$$

In the above, ζ_3 is a cubic root of 1. If P is a point of $L_{3n}(2)$ and Q is a point of M_n,

$$\langle P, Q\rangle = \langle \sigma_2(P), \sigma_2(Q)\rangle = \langle P, -Q\rangle = -\langle P, Q\rangle\,.$$

Then we have $\langle P, Q\rangle = 0$.

If P is a point of $L_{2n}(3)$ and $Q = (x, y)$ is a point of M_n,

$$\begin{aligned}\langle P, Q\rangle &= \langle \sigma_3(P), \sigma_3(Q)\rangle = \langle P, \zeta_3 Q\rangle \\ &= \langle \sigma_3^2(P), \sigma_3^2(Q)\rangle = \langle P, \zeta_3^2 Q\rangle\,.\end{aligned}$$

Here $\zeta_3 Q = (\zeta_3 x, y)$ and $\zeta_3^2 Q = (\zeta_3^2 x, y)$. Since $Q + \zeta_3 Q + \zeta_3^2 Q = O$, we have $\langle P, Q\rangle = 0$. This means that $L_{2n}(3) + L_{3n}(2)$ and M_n are orthogonal. *q.e.d.*

Let C_2 be the elliptic curve defined by

$$C_2 : y^2 = x^3 + t^3 - t\,.$$

In the same way as $E^{(3)}$, we study C_2 following [S4].

The associated elliptic surface has a singular fibre of type I_0^* at $t = \infty$. $C_2(k(t)) \cong D_4^*$ ([O-S, Main Theorem No. 9]).

The points of the form:

$$\begin{cases} x = g + a(t-1) \\ y = h + c(t-1) \end{cases} \quad g, a, h, c \in k\,. \tag{3.3}$$

generate $C_2(k(t))$. We substitute (3.3) into the defining equation of C_2, and look at the coefficients of $(t-1)^m$ for $m = 0, 1, 2, 3$. Then we have

$$h^2 = g^3 \tag{3.4}$$

$$2hc = 3g^2 a + 2 \tag{3.5}$$

$$c^2 = 3ga^2 + 3 \tag{3.6}$$

$$0 = a^3 + 1 \tag{3.7}$$

The points such that $a = -1$ generate a sublattice $1_4'$ which is isomorphic to the unit matrix of degree four. Let $u = \mathrm{sp}_1(P) = g/h$. By (3.4), we have $g = u^{-2}$, $h = u^{-3}$. By (3.5) and (3.6), we have

$$4u^8 - 12u^4 - 12u^2 - 3 = 0. \tag{3.8}$$

For roots u of (3.8), letting $g = u^{-2}$, $h = u^{-3}$ and $c = (-3g^2 + 2)/2h$, we have all points of the form:

$$\begin{cases} x = g - (t-1) \\ y = h + c(t-1) \end{cases} \tag{3.9}$$

If $(x(t), y(t))$ is a $k(t)$-rational point of C_2, then $(x(t^6)/t^2, y(t^6)/t^3)$ is a $k(t)$-rational point of $E^{(12)}$. So

$$M_2 = \{(x(t^6)/t^2, y(t^6)/t^3) \mid (x(t), y(t)) \in C_2(k(t))\}$$

is a sublattice of L_{12}. We have $M_2 \cong D_4^*(6)$.

LEMMA 3.2. *$L_4(3) + L_6(2) + M_2$ is a sublattice of finite index in L_{12}.*

Proof. By Lemma 3.1, $L_4(3) + L_6(2)$ and M_2 are orthogonal. The rank of $L_4(3) + L_6(2)$ is $l_4 + l_6 - l_2 = 6 + 8 - 2 = 12$ and the rank of M_2 is 4. Since the rank of L_{12} is $l_{12} = 16$, $L_4(3) + L_6(2) + M_2$ is a sublattice of finite index in L_{12}. *q.e.d.*

REMARK 2. If $P = (x(t), y(t)) \in L_{12}$, then $\tilde{P} = (-t^4 x(1/t), \sqrt{-1}t^6 y(1/t)) \in L_{12}$. For a sublattice N of L_{12}, we denote

$$\tilde{N} = \{\tilde{P} \mid P \in N\}.$$

The sublattice $L_4(3) + L_6(2) + \widetilde{L_4(3)}$ includes the sublattice $L_4(3) + L_6(2) + M_2$.

$E^{(4)}$ is the (E_6, q_0)-model in [S7]. In this case $q_0 = -1$. The universal polynomial is given in [S7]. So we have the fundamental algebraic equation:

$$u^3(u^{24} - 17280u^{12} - 110592) = 0. \tag{3.10}$$

For L_6, the splitting field is given in [S6, Theorem 1]. It is $\mathbb{Q}(\zeta_{12}, \sqrt[3]{2})$. So if the equations

$$u^{12} - 1 = 0 \quad \text{and} \quad u^3 - 2 = 0 \tag{3.11}$$

can be solved in $\mathbb{F}_p$, then we have $L_6^{(p)} = E^{(6)}(\mathbb{F}_p(t))$.

The minimal prime number p such that the equations (3.8), (3.10) and (3.11) can be solved in $\mathbb{F}_p$ is 397. We have the following Proposition:

PROPOSITION 3. $L_{12}^{(397)} := E^{(12)}(\overline{\mathbb{F}_{397}}(t)) = E^{(12)}(\mathbb{F}_{397}(t))$.

Proof. Let $p = 397$. In this case, $d = 6m/(m, 6) = 12$. Since $p \equiv 1 \bmod 12$, by the algorithm of [S1], we have the rank of $L_{12}^{(p)}$ is equal to $l_{12} = 16$.

We use notation $L_4^{(p)}(3)$, $L_6^{(p)}(2)$ and $M_2^{(p)}$ as in the proof of Proposition 1. Then $L_4^{(p)}(3) + L_6^{(p)}(2) + M_2^{(p)}$ is a sublattice of finite index in $L_{12}^{(p)}$.

All the roots of (3.10) belong to $\mathbb{F}_p$. By Theorem 1, the points of the form:

$$(x_i(t), y_i(t)) = (a_i t + b_i, t^2 + d_i t + e_i) \quad (i = 1, \dots, 6)$$

generate $L_4^{(p)}$ and $a_i, b_i, d_i, e_i \in \mathbb{F}_p$. Then the points

$$(x_i(t^3), y_i(t^3)) \quad (i = 1, \dots, 6)$$

generate $L_4^{(p)}(3)$. These points belong to $E^{(12)}(\mathbb{F}_p(t))$. So we have $L_4^{(p)}(3) \subset E^{(12)}(\mathbb{F}_p(t))$.

Since the equations (3.11) can be solved in $\mathbb{F}_p$, we have $L_6^{(p)} = E^{(6)}(\mathbb{F}_p(t))$. So we have $L_6^{(p)}(2) \subset E^{(12)}(\mathbb{F}_p(t))$.

All the roots u of (3.8) belong to $\mathbb{F}_p$. Letting $g = u^{-2}$, $h = u^{-3}$ and $c = (-3g^2 + 3)/2h$, we have the points of the form:

$$(g - (t-1), h + c(t-1)).$$

They generate the subgroup $1_4'$ of index 2 in $C_2(\overline{\mathbb{F}_p}(t))$. Then the points

$$((g - (t^6 - 1))/t^2, (h + c(t^6 - 1))/t^3)$$

generate a subgroup of index 2 in $M_2^{(p)}$. These points belong to $E^{(12)}(\mathbb{F}_p(t))$.

By Lemma 2.1, we have $E^{(12)}(\mathbb{F}_p(t)) = E^{(12)}(\overline{\mathbb{F}_p}(t))$. *q.e.d.*

By (3.2), the minimal number of $\langle P, P\rangle$ is 4 and

$$\langle P, P\rangle = 4 \Leftrightarrow (PO) = 0.$$

The points $P \in L_{12}^{(397)}$ such that $\langle P, P\rangle = 4$ are called minimal sections. By the same argument as [S2, Lemma 10.9], the minimal sections are of the form:

$$(3.12) \qquad \begin{cases} x = x_0 + x_1(t-1) + \cdots + x_4(t-1)^4 \\ y = y_0 + y_1(t-1) + \cdots + y_6(t-1)^6 \end{cases} \quad x_0, \dots, x_4, y_0, \dots, y_6 \in \mathbb{F}_{397}.$$

We substitute (3.12) into the defining equation of $E^{(12)}$, and look at the coefficients of $(t-1)^m$ for $m = 0, 1, 2, \dots, 12$. Then we have

$$(3.13) \qquad y_0^2 = x_0^3$$

$$(3.14) \qquad 2y_0y_1 = 12 + 3x_0^2x_1$$

$$(3.15) \qquad y_1^2 + 2y_0y_2 = 66 + 3x_0x_1^2 + 3x_0^2x_2$$

$$(3.16) \qquad 2y_1y_2 + 2y_0y_3 = 220 + x_1^3 + 6x_0x_1x_2 + 3x_0^2x_3$$

$$(3.17) \qquad y_2^2 + 2y_1y_3 + 2y_0y_4 = 495 + 3x_1^2x_2 + 3x_0x_2^2 + 6x_0x_1x_3 + 3x_0^2x_4$$

$$(3.18) \qquad 2y_2y_3 + 2y_1y_4 + 2y_0y_5 = 792 + 3x_1x_2^2 + 3x_1^2x_3 + 6x_0x_2x_3 + 6x_0x_1x_4$$

$$(3.19) \qquad 2y_6y_0 + y_3^2 + 2y_2y_4 + 2y_1y_5 = 924 + x_2^3 + 6x_1x_2x_3 + 3x_0x_3^2 + 3x_1^2x_4 + 6x_0x_2x_4$$

$$(3.20) \qquad 2y_6y_1 + 2y_3y_4 + 2y_2y_5 = 792 + 3x_2^2x_3 + 3x_1x_3^2 + 6x_1x_2x_4 + 6x_0x_3x_4$$

$$2y_6y_2 + y_4^2 + 2y_3y_5 = 495 + 3x_2x_3^2 + 3x_2^2x_4 + 6x_1x_3x_4 + 3x_0x_4^2 \tag{3.21}$$

$$2y_6y_3 + 2y_4y_5 = 220 + x_3^3 + 6x_2x_3x_4 + 3x_1x_4^2 \tag{3.22}$$

$$2y_6y_4 + y_5^2 = 66 + 3x_3^2x_4 + 3x_2x_4^2 \tag{3.23}$$

$$2y_6y_5 = 12 + 3x_3x_4^2 \tag{3.24}$$

$$y_6^2 = 1 + x_4^3 \tag{3.25}$$

Let $u = \mathrm{sp}_1(P) = x_0/y_0$. By (3.13), we have $x_0 = u^{-2}$ and $y_0 = u^{-3}$. By (3.14), ..., (3.19), $y_1, \ldots, y_6$ are polynomials of u, u^{-1}, x_1, x_2, x_3 and x_4. We substitute every element of $\mathbb{F}_{397}^{\times}$ into u and every element of $\mathbb{F}_{397}$ into x_1 and x_2. By (3.20), we have an equation

$$(x_3 - a)x_4 = f(x_3) \quad a \in \mathbb{F}_{397} \text{ and } f(x_3) \in \mathbb{F}_{397}[x_3].$$

If $f(a) = 0$, then we substitute a into x_3, and check if there is x_4 satisfying the equations (3.21), ..., (3.25). If $f(a) \neq 0$, then we substitute $f(x_3)/(x_3 - a)$ into x_4, and check if there is x_3 satisfying the equations (3.21), ..., (3.25). Then we can get all minimal sections.

In this way, we find 1848 minimal sections. Let $L_{12\,\mathrm{min}}^{(397)}$ be the lattice generated by these 1848 minimal sections.

For two minimal sections $P = (x(t), y(t))$ and $Q = (z(t), w(t))$, by (3.1), we have

$$\langle P, Q\rangle = 2 - (PQ).$$

The intersection number (PQ) is calculated by

$$\begin{aligned}(PQ) &= \deg(\gcd(x(t) - z(t), y(t) - w(t))) \\ &\quad + \min(4 - \deg(x(t) - z(t)), 6 - \deg(y(t) - w(t))).\end{aligned}$$

Here $\min(a, b)$ is the smaller number of a and b, $\deg(f(t))$ is the degree of polynomial $f(t)$ and $\gcd(f(t), g(t))$ is the greatest common divisor of two polynomials $f(t)$ and $g(t)$. The second term $\min(4 - \deg(x(t) - z(t)), 6 - \deg(y(t) - w(t)))$ is the intersection multiplicity at $t = \infty$.

In $L_6(2)$, $L_4(3)$ and $\widetilde{L_4(3)}$ (see Remark 2), we can find independent 16 minimal sections. Taking the height pairings with them, we can treat all sections as lattice points in $\mathbb{Z}^{16}$. Then, by elimination, we can find generators.

PROPOSITION 4. *$L_{12\,\mathrm{min}}^{(397)}$ has the following properties:*
There are 1848 *minimal sections.*
The following 16 *points generate $L_{12\,\mathrm{min}}^{(397)}$:*

$$P_1 = (1, t^6)$$

$$P_2 = (34, t^6)$$

$$P_3 = (396t^4, 63)$$

$$P_4 = (363t^4, 63)$$

$$P_5 = (367t^4 + 362, 328t^6 + 236t^2)$$
$$P_6 = (363t^4 + 141, 179t^4 + 20)$$
$$P_7 = (2t^4 + 266t^2 + 30, 394t^6 + 262t^4 + 307t^2 + 377)$$
$$P_8 = (171t^4 + 87t^2 + 329, 328t^6 + 285t^4 + 229t^2 + 208)$$
$$P_9 = (207t^3 + 59, t^6 + 183t^3 + 319)$$
$$P_{10} = (289t^3 + 21, t^6 + 183t^3 + 319)$$
$$P_{11} = (338t^4 + 190t, 247t^6 + 16t^3 + 63)$$
$$P_{12} = (376t^4 + 108t, 247t^6 + 16t^3 + 63)$$
$$P_{13} = (338t^4 + 60t, 247t^6 + 214t^3 + 334)$$
$$P_{14} = (342t^3 + 21, t^6 + 381t^3 + 78)$$
$$P_{15} = (224t^4 + 17t^3 + 297t^2 + 230t + 324, \\ 281t^6 + 127t^5 + 217t^4 + 207t^3 + 297t^2 + 49t + 162)$$
$$P_{16} = (73t^4 + 167t^3 + 100t^2 + 380t + 173, \\ 281t^6 + 308t^5 + 52t^4 + 337t^3 + 173t^2 + 61t + 235)$$

The Gram matrix is

$$\begin{pmatrix}
4 & -2 & 0 & 0 & 2 & 0 & 0 & -2 & 1 & -2 & 0 & 0 & 0 & -2 & -1 & -1 \\
-2 & 4 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & -1 & 2 \\
0 & 0 & 4 & -2 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & -2 & -1 & 0 & 1 & -1 \\
0 & 0 & -2 & 4 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & -2 & -1 \\
2 & -2 & 0 & 0 & 4 & 0 & 2 & -2 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & -2 & 0 & 0 & 4 & 2 & -2 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & -2 & 2 & 2 & 4 & -2 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 1 \\
-2 & 0 & 0 & 0 & -2 & -2 & -2 & 4 & -1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -2 & 0 & 0 & 0 & -2 & 0 & 1 \\
-2 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & -2 & 4 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 4 & -2 & -1 & 0 & -1 & 0 \\
0 & 0 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 4 & -1 & 0 & -1 & 0 \\
0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & 4 & 0 & 1 & 2 \\
-2 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 4 & 1 & 1 \\
-1 & -1 & 1 & -2 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 1 & 1 & 4 & 0 \\
-1 & 2 & -1 & -1 & -1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 2 & 1 & 0 & 4
\end{pmatrix}$$

The determinant is $1296 = 2^4 3^4$.
The center density (*cf.* [C-S]) *is* 1/36.

We have the following theorem.

THEOREM 3. $L_{12}^{(397)} = L_{12\,\min}^{(397)}$.

So $L_{12}^{(397)}$ has the properties given in Proposition 4.

Proof. Since $L_{12}^{(397)}$ is an integral lattice, the determinant of $L_{12}^{(397)}$ is a divisor of 1296. So if $L_{12}^{(397)} \neq L_{12\min}^{(397)}$, there is a point $P \in L_{12}^{(397)}$ such that $P \notin L_{12\min}^{(397)}$ and $2P$ or $3P \in L_{12\min}^{(397)}$.

Let P be a point in $L_{12}^{(397)}$ such that $P \notin L_{12\min}^{(397)}$ and $3P \in L_{12\min}^{(397)}$. We can assume that

$$3P = a_1 P_1 + \cdots + a_{16} P_{16} \quad (a_i = -1, 0, 1)\,.$$

We can calculate

$$\begin{aligned}\langle P, P_i\rangle &= \frac{1}{3}\langle 3P, P_i\rangle = \frac{1}{3}\langle a_1 P_1 + \cdots + a_{16} P_{16}, P_i\rangle \\ &= \frac{1}{3}(a_1\langle P_1, P_i\rangle + \cdots + a_{16}\langle P_{16}, P_i\rangle)\,.\end{aligned}$$

It must be an integer. Solving the simultaneous equations

$$a_1 \cdot \langle P_1, P_i\rangle + \cdots + a_{16} \cdot \langle P_{16}, P_i\rangle = 0 \quad (i = 1, 2, \ldots, 16)$$

in $\mathbb{F}_3$, we get the following relations modulo 3:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_9 \\ a_{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 2 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{10} \\ a_{12} \\ a_{13} \\ a_{14} \end{pmatrix},$$

$$a_5 = a_6 = a_7 = a_8 = a_{15} = a_{16} = 0\,.$$

We substitute -1, 0, 1 into a_{10}, a_{12}, a_{13} and a_{14}, and calculate

$$\langle P, P\rangle = \frac{1}{9}\langle 3P, 3P\rangle = \cdots\,.$$

$\langle P, P\rangle$ must be an even number greater than 4, but there is no such case. So there is no P in $L_{12}^{(397)}$ such that $P \notin L_{12\min}^{(397)}$ and $3P \in L_{12\min}^{(397)}$.

Next let P be a point in $L_{12}^{(397)}$ such that $P \notin L_{12\min}^{(397)}$ and $2P \in L_{12\min}^{(397)}$. We can assume that

$$2P = a_1 P_1 + \cdots + a_{16} P_{16} \quad (a_i = 0, 1)\,.$$

We can calculate

$$\begin{aligned}\langle P, P_i\rangle &= \frac{1}{2}\langle 2P, P_i\rangle = \frac{1}{2}\langle a_1 P_1 + \cdots + a_{16} P_{16}, P_i\rangle \\ &= \frac{1}{2}(a_1\langle P_1, P_i\rangle + \cdots + a_{16}\langle P_{16}, P_i\rangle)\,.\end{aligned}$$

Solving the simultaneous equations

$$a_1 \cdot \langle P_1, P_i\rangle + \cdots + a_{16} \cdot \langle P_{16}, P_i\rangle = 0 \quad (i = 1, 2, \ldots, 16)$$

in $\mathbb{F}_2$, we get the following relations modulo 2:

$$a_1 = a_5\,, \quad a_2 = a_5 + a_8\,, \quad a_3 = a_7\,, \quad a_4 = a_6\,,$$

$$a_9 = a_{10} = a_{11} = a_{12} = a_{13} = a_{14} = a_{15} = a_{16} = 0\,.$$

We substitute 0 or 1 into a_5, a_6, a_7 and a_8, and calculate

$$\langle P, P\rangle = \frac{1}{4}\langle 2P, 2P\rangle = \cdots\,.$$

$\langle P, P\rangle$ must be an even number greater than 4, but there is no such case. So there is no P in $L_{12}^{(397)}$ such that $P \notin L_{12\,\min}^{(397)}$ and $2P \in L_{12\,\min}^{(397)}$. *q.e.d.*

COROLLARY 2. *If we assume* (∗), *then* L_{12} *has the following properties*:
There are 1848 *minimal sections of norm* 4.
The Gram matrix is the same as in Proposition 4.
The determinant is 1296.
The center density is 1/36.

4. The lattices L_{18}, L_{24}, L_{30} and L_{60}

About the four lattices L_{18}, L_{24}, L_{30} and L_{60}, we do not yet know the lattice structures. In this section, we give splitting prime numbers of L_{18}, L_{24} and L_{30}. Unfortunately, we can not give a splitting prime number of L_{60}.

4.1. L_{18}

Let C_3 be the elliptic curve defined by

$$C_3 : y^2 = x^3 + t^4 - t\,.$$

The fundamental algebraic equation is given by (2.4). The lattice

$$M_3 = \{(x(t^6)/t^2, y(t^6)/t^3) \mid (x(t), y(t)) \in C_3(k(t))\}$$

is a sublattice of L_{18}. We have $M_3 \cong E_6^*(6)$.

LEMMA 4.1. $L_6(3) + L_9(2) + M_3$ *is a sublattice of finite index in* L_{18}.

Proof. By Lemma 3.1, $L_6(3) + L_9(2)$ and M_3 are orthogonal. The rank of $L_6(3) + L_9(2)$ is $l_6 + l_9 - l_3 = 8 + 10 - 4 = 14$ and the rank of M_3 is 6. Since the rank of L_{18} is $l_{18} = 20$, $L_6(3) + L_9(2) + M_3$ is a sublattice of finite index in L_{18}. *q.e.d.*

The minimal prime number p such that the equations (2.4), (2.10) and (3.11) can be solved in $\mathbb{F}_p$ is 433. We have the following Proposition:

PROPOSITION 5. $L_{18}^{(433)} := E^{(18)}(\overline{\mathbb{F}_{433}}(t)) = E^{(18)}(\mathbb{F}_{433}(t))$.

Proof. Let $p = 433$. In this case, $d = 6m/(m, 6) = 18$. Since $p \equiv 1 \bmod 18$, by the algorithm of [S1], we have the rank of $L_{18}^{(p)}$ is equal to $l_{18} = 20$.

We use notation $L_6^{(p)}(3)$, $L_9^{(p)}(2)$ and $M_3^{(p)}$ as in the proof of Proposition 1. Then $L_6^{(p)}(3) + L_9^{(p)}(2) + M_3^{(p)}$ is a sublattice of finite index in $L_{18}^{(p)}$.

By Proposition 1, we have $L_9^{(p)} = E^{(9)}(\mathbb{F}_p(t))$. So we have $L_9^{(p)}(2) \subset E^{(18)}(\mathbb{F}_p(t))$.

Since the equations (3.11) can be solved in $\mathbb{F}_p$, we have $L_6^{(p)} = E^{(6)}(\mathbb{F}_p(t))$. So we have $L_6^{(p)}(3) \subset E^{(18)}(\mathbb{F}_p(t))$.

All the roots of (2.4) belong to $\mathbb{F}_p$. By Theorem 1, the points of the form:

$$(x_i(t), y_i(t)) = (a_i t + b_i, t^2 + d_i t + e_i) \quad (i = 1, \ldots, 6)$$

generate $C_3(\overline{\mathbb{F}_p}(t))$ and $a_i, b_i, d_i, e_i \in \mathbb{F}_p$. Then the points

$$(x_i(t^6)/t^2, y_i(t^6)/t^3) \quad (i = 1, \ldots, 6)$$

generate $M_3^{(p)}$. These points belong to $E^{(18)}(\mathbb{F}_p(t))$. So we have $M_3^{(p)} \subset E^{(18)}(\mathbb{F}_p(t))$.

By Lemma 2.1, we have $E^{(18)}(\mathbb{F}_p(t)) = E^{(18)}(\overline{\mathbb{F}_p}(t))$. *q.e.d.*

REMARK 3. If $P = (x(t), y(t)) \in L_{18}$, then $\tilde{P} = (-t^6 x(1/t), \sqrt{-1}t^9 y(1/t)) \in L_{18}$. For a sublattice N of L_{18}, we denote

$$\tilde{N} = \{\tilde{P} \mid P \in N\}.$$

The sublattice $L_6(3) + L_9(2) + \widetilde{L_9(2)}$ includes the sublattice $L_6(3) + L_9(2) + M_3$.

4.2. L_{24}

Let C_4 be the elliptic curve defined by

$$C_4 : y^2 = x^3 + t^5 - t.$$

C_4 is the (E_8, q_1)-model in [S7]. In this case $q_1 = -1$. The universal polynomial is given in [S7]. So we have the fundamental algebraic equation:

$$u^{240} - 419237280u^{216} + \cdots = 0. \tag{4.1}$$

The lattice

$$M_4 = \{(x(t^6)/t^2, y(t^6)/t^3) \mid (x(t), y(t)) \in C_4(k(t))\}$$

is a sublattice of L_{24}. We have $M_4 \cong E_8(6)$.

LEMMA 4.2. *$L_{12}(2) + M_4$ is a sublattice of finite index in L_{24}.*

Proof. By Lemma 3.1, $L_{12}(2)$ and M_4 are orthogonal. The rank of $L_{12}(2)$ is $l_{12} = 16$ and the rank of M_4 is 8. Since the rank of L_{24} is $l_{24} = 24$, $L_{12}(2) + M_4$ is a sublattice of finite index in L_{24}. *q.e.d.*

The minimal prime number p such that the equations (3.8), (3.10), (3.11) and (4.1) can be solved in $\mathbb{F}_p$ is 1801. We have the following Proposition:

PROPOSITION 6. $L_{24}^{(1801)} := E^{(24)}(\overline{\mathbb{F}_{1801}}(t)) = E^{(24)}(\mathbb{F}_{1801}(t))$.

Proof. Let $p = 1801$. In this case, $d = 6m/(m, 6) = 24$. Since $p \equiv 1 \bmod 24$, by the algorithm of [S1], we have the rank of $L_{24}^{(p)}$ is equal to $l_{24} = 24$.

We use notation $L_{12}^{(p)}(2)$ and $M_4^{(p)}$ as in the proof of Proposition 1. Then $L_{12}^{(p)}(2) + M_4^{(p)}$ is a sublattice of finite index in $L_{24}^{(p)}$.

The equation (3.8), (3.10) and (3.11) can be solved in $\mathbb{F}_p$. In the same way as the proof of Proposition 4, we can show that $L_{12}^{(p)} = E^{(12)}(\mathbb{F}_p(t))$. So we have $L_{12}^{(p)}(2) \subset E^{(24)}(\mathbb{F}_p(t))$.

All the roots of (4.1) belong to $\mathbb{F}_p$. By Theorem 1, the points of the form:

$$(x_i(t), y_i(t)) = (g_i t^2 + a_i t + b_i, h_i t^3 + c_i t^2 + d_i t + e_i) \quad (i = 1, \dots, 8)$$

generate $C_4(\overline{\mathbb{F}_p}(t))$ and $g_i, a_i, b_i, h_i, c_i, d_i, e_i \in \mathbb{F}_p$. Then the points

$$(x_i(t^6)/t^2, y_i(t^6)/t^3) \quad (i = 1, \dots, 8)$$

generate $M_4^{(p)}$. These points belong to $E^{(24)}(\mathbb{F}_p(t))$. So we have $M_4^{(p)} \subset E^{(24)}(\mathbb{F}_p(t))$.

By Lemma 2.1, we have $E^{(24)}(\mathbb{F}_p(t)) = E^{(24)}(\overline{\mathbb{F}_p}(t))$. *q.e.d.*

4.3. L_{30}

Let C_5 be the elliptic curve defined by

$$C_5 : y^2 = x^3 + t^6 - t\,.$$

Then

$$M_5 = \{(x(t^6)/t^2, y(t^6)/t^3) \mid (x(t), y(t)) \in C_5(k(t))\}$$

is a sublattice of L_{30}. But M_5 can be seen in a different way:

$$M_5 = \{(-t^{10}x(1/t^6), \sqrt{-1}t^{15}y(1/t^6) \mid (x(t), y(t)) \in E^{(5)}(k(t))\}\,.$$

$E^{(5)}$ is the (E_8, q_0)-model in [S7]. In this case $q_0 = -1$. The universal polynomial is given in [S7]. So we have the fundamental algebraic equation:

$$u^{240} - 65945880000u^{210} + \cdots = 0\,. \tag{4.2}$$

We have $M_5 \cong E_8(6)$.

LEMMA 4.3. *$L_6(5) + L_5(6) + M_5$ is a sublattice of finite index in L_{30}.*

Proof. Since $L_5(6)$ is a sublattice of $L_{15}(2)$, by Lemma 3.1, $L_5(6)$ and M_5 are orthogonal.

Next we show that $L_6(5) \cap M_5 = \{O\}$. We define $\sigma_5 \in \mathrm{Gal}(k(t)/k(t^5))$ by

$$\sigma_5 : t \to \zeta_5 t\,.$$

In the above, ζ_5 is a 5-th root of 1.

Let $P = (x(t^6)/t^2, y(t^6)/t^3)$ be a point of $L_6(5) \cap M_5$. Since $P = P^{\sigma_5} = (x(\zeta_5 t^6)/\zeta_5^2 t^2, y(\zeta_5 t^6)/\zeta_5^3 t^3)$, we have

$$\begin{aligned} x(\zeta_5 t^6) &= \zeta_5^2 x(t^6)\,, & y(\zeta_5 t^6) &= \zeta_5^3 y(t^6)\,, \\ x(\zeta_5 t) &= \zeta_5^2 x(t), & y(\zeta_5 t) &= \zeta_5^3 y(t)\,, \\ x(\zeta_5 t)/\zeta_5^2 t^2 &= x(t)/t^2\,, & y(\zeta_5 t)/\zeta_5^3 t^3 &= y(t)/t^3\,. \end{aligned}$$

This means that $x(t)/t^2$ and $y(t)/t^3$ are σ_5-invariant. So there are rational functions $x_1(t)$ and $y_1(t)$ such that $x(t)/t^2 = x_1(t^5)$ and $y(t)/t^3 = y_1(t^5)$. Substituting them into $y(t)^2 =$

$x(t)^3 + t^6 - t$, we have

$$t^6 y_1(t^5)^2 = t^6 x_1(t^5)^3 + t^6 - t\,,$$
$$t^5 y_1(t^5)^2 = t^5 x_1(t^5)^3 + t^5 - 1\,,$$
$$t y_1(t)^2 = t x_1(t)^3 + t - 1\,,$$
$$y_1(t)^2 = x_1(t)^3 + 1 - 1/t\,,$$
$$y_1(1/t)^2 = x_1(1/t)^3 + 1 - t\,.$$

The associated elliptic surface has a singular fibre of type II* at $t = \infty$. By the theory of the Mordell-Weil lattices of rational elliptic surfaces, this elliptic curve has only one rational point O ([O-S, Main Theorem No. 62]). So we have $L_6(5) \cap M_5 = \{O\}$.

The rank of $L_6(5) + L_5(6)$ is $l_6 + l_5 - l_1 = 8 + 8 - 0 = 16$ and the rank of M_6 is 8. Since the rank of L_{30} is $l_{30} = 24$, $L_6(5) + L_5(6) + M_5$ is a sublattice of finite index in L_{30}. *q.e.d.*

The minimal prime number p such that the equations (3.11) and (4.2) can be solved in $\mathbb{F}_p$ is 25261. We have the following Proposition:

PROPOSITION 7. $L_{30}^{(25261)} := E^{(30)}(\overline{\mathbb{F}_{25261}}(t)) = E^{(30)}(\mathbb{F}_{25261}(t))$.

Proof. Let $p = 25261$. $\sqrt{-1} = 3086$ in $\mathbb{F}_p$. In this case, $d = 6m/(m, 6) = 30$. Since $p \equiv 1 \bmod 30$, by the algorithm of [S1], we have the rank of $L_{30}^{(p)}$ is equal to $l_{30} = 24$.

We use notation $L_6^{(p)}(5)$, $L_5^{(p)}(6)$ and $M_5^{(p)}$ as in the proof of Proposition 1. Then $L_6^{(p)}(5) + L_5^{(p)}(6) + M_5^{(p)}$ is a sublattice of finite index in $L_{30}^{(p)}$.

Since the equations (3.11) can be solved in $\mathbb{F}_p$, we have $L_6^{(p)} = E^{(6)}(\mathbb{F}_p(t))$. So we have $L_6^{(p)}(5) \subset E^{(30)}(\mathbb{F}_p(t))$.

All the roots of (4.2) belong to $\mathbb{F}_p$. By Theorem 1, the points of the form:

$$(x_i(t), y_i(t)) = (g_i t^2 + a_i t + b_i, h_i t^3 + c_i t^2 + d_i t + e_i) \quad (i = 1, \dots, 8)$$

generate $L_5^{(p)}$ and $g_i, a_i, b_i, h_i, c_i, d_i, e_i \in \mathbb{F}_p$. Then the points

$$(x_i(t^6), y_i(t^6)) \quad (i = 1, \dots, 8)$$

generate $L_5^{(p)}(6)$ and the points

$$(-t^{10} x_i(1/t^6), \sqrt{-1} t^{15} y_i(1/t^6)) \quad (i = 1, \dots, 8)$$

generate $M_5^{(p)}$. These points belong to $E^{(30)}(\mathbb{F}_p(t))$. So we have $L_5^{(p)}(6) \subset E^{(30)}(\mathbb{F}_p(t))$ and $M_5^{(p)} \subset E^{(30)}(\mathbb{F}_p(t))$.

By Lemma 2.1, we have $E^{(30)}(\mathbb{F}_p(t)) = E^{(30)}(\overline{\mathbb{F}_p}(t))$. *q.e.d.*

4.4. L_{60}

Let C_{10} be the elliptic curve defined by

$$C_{10} : y^2 = x^3 + t^{11} - t\,.$$

Then

$$M_{10} = \{(x(t^6)/t^2, y(t^6)/t^3) \mid (x(t), y(t)) \in C_{10}(k(t))\}$$

is a sublattice of L_{60}. Calculating by the algorithm of [S1], we have that the rank of C_{10} is 16. So the rank of M_{10} is 16.

LEMMA 4.4. *$L_{12}(5) + L_{30}(2) + M_{10}$ is a sublattice of finite index in L_{60}.*

Proof. By Lemma 3.1, $L_{30}(2)$ and M_{10} are orthogonal.

Let $P = (x(t^6)/t^2, y(t^6)/t^3)$ be a point of $L_{12}(5) \cap M_{10}$. As in the proof of Lemma 4.3, we can show that there are rational functions $x_1(t)$ and $y_1(t)$ such that $x(t)/t^2 = x_1(t^5)$ and $y(t)/t^3 = y_1(t^5)$. Substituting them into $y(t)^2 = x(t)^3 + t^{11} - t$, we have

$$\begin{aligned} t^6 y_1(t^5)^2 &= t^6 x_1(t^5)^3 + t^{11} - t\,, \\ t^5 y_1(t^5)^2 &= t^5 x_1(t^5)^3 + t^{10} - 1\,, \\ t y_1(t)^2 &= t x_1(t)^3 + t^2 - 1\,, \\ (t^3 y_1(t))^2 &= (t^2 x_1(t))^3 + t^7 - t^5\,. \end{aligned}$$

Calculating by the algorithm of [S1], we have that the rank of this elliptic curve is 0. So we have $L_{12}(5) \cap M_{10} = \{O\}$.

The rank of $L_{12}(5) + L_{30}(2)$ is $l_{12} + l_{30} - l_6 = 16 + 24 - 8 = 32$ and the rank of M_{10} is 16. Since the rank of L_{60} is $l_{60} = 48$, $L_{12}(5) + L_{30}(2) + M_{10}$ is a sublattice of finite index in L_{60}. *q.e.d.*

The Kodaira-Néron model of C_{10} is not a rational surface and we do not have the universal polynomial. So we can not find a splitting prime number of $E^{(60)}$ in the same way as the other cases. But rank 8 part of C_{10} come from rational case ([CMT]).

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Department of Mathematics
Gunma College of Technology
580 Toriba, Maebashi, Gunma 371
Japan
E-mail: usui@nat.gunma-ct.ac.jp